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THREE KINDS OF NUMERICAL INDICES OF A BANACH SPACE II

Sung Guen Kim*

Department of Mathematics, Kyungpook National University, Daegu 702-701, Republic of Korea.
E-Mail sgk317@knu.ac.kr

Abstract. For a Banach space E and a positive integer k, we study about three kinds of numerical indices of E, the multilinear numerical index $n_m^{(k)}(E)$, the symmetric multilinear numerical index $n_s^{(k)}(E)$ and the polynomial numerical index $n_p^{(k)}(E)$. First we show that $n_I^{(k)}(E^{**}) \leq n_I^{(k)}(E)$ for $I = m, s$ and present some inequalities among $n_m^{(k)}(E), n_s^{(k)}(E)$ and $n_p^{(k)}(E)$. We also prove that if E is a strictly convex Banach space, then $n_m^{(k)}(E) = 0$ for every $k \geq 2$.

Mathematics Subject Classification (2010): Primary 46A22, 46G20; Secondary 46G25.
Key words: Numerical radius, numerical index, multilinear mappings, symmetric multilinear mappings, homogeneous polynomials.

1. Introduction. Throughout this paper $\mathbb{K}$ denotes either the complex field $\mathbb{C}$ or the real field $\mathbb{R}$. If the field is not specified the results are valid in both cases. Let E and F be Banach spaces over the field $\mathbb{K}$. We write B_E and S_E for the closed unit ball and the unit sphere of E, respectively. The dual space of E is denoted by E^*. We write E^k for the product $E \times \cdots \times E$ with k factors, for some natural number k. We denote by $\mathcal{L}(^kE : F)$ the Banach space of continuous k-linear mappings of E^k into F endowed with the norm

$$\|A\| = \sup \ \{\|A(x_1, \ldots, x_k)\| \ : x_j \in B_E, j = 1, \ldots, k\}.$$

$A \in \mathcal{L}(^kE : F)$ is said to be symmetric if

$$A(x_1, \cdots, x_k) = A(x_{\sigma(1)}, \cdots, x_{\sigma(k)})$$

for any $x_1, \cdots, x_k$ in E and any permutation σ of the first k natural numbers. We denote by $\mathcal{L}_s(^kE : F)$ the closed subspace of all symmetric k-linear maps in $\mathcal{L}(^kE : F)$. We define the symmetric $k-$linear mapping $A_s : E^k \to F$ (which we call the *symmetrization* of A) by

$$A_s(x_1, \cdots, x_k) = \frac{1}{k!} \sum_{\sigma} A(x_{\sigma(1)}, \cdots, x_{\sigma(k)})$$

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for any $x_1, \cdots, x_k$ in E, where the summation is over the $k!$ permutations σ of the first k natural numbers. We denote $\mathcal{L}(^kE : \mathbb{K})$ and $\mathcal{L}_s(^kE : \mathbb{K})$ by $\mathcal{L}(^kE)$ and $\mathcal{L}_s(^kE)$ respectively. A mapping $P : E \to F$ is said to be a continuous k-homogeneous polynomial if there exists an $A \in \mathcal{L}(^kE : F)$ such that $P(x) = A(x, \cdots, x)$ for all $x \in E$. For $A \in \mathcal{L}(^kE : F)$, we define the associated polynomial $\widehat{A} : E \to F$ by $\widehat{A}(x) = A(x, \cdots, x)$ for $x \in E$. It is obvious that $\widehat{A} = \widehat{A_s}$. We denote by $\mathcal{P}(^kE : F)$ the Banach space of continuous k-homogeneous polynomials of E into F endowed with the polynomial norm $\|P\| = \sup_{x \in B_E} \|P(x)\|$. We denote $\mathcal{P}(^kE : \mathbb{K})$ by $\mathcal{P}(^kE)$. We also note that $\|\hat{A}\| \leq \|A_s\| \leq \|A\|$ for any A in $\mathcal{L}(^kE : F)$. See [D] for a general background on the theory of polynomials on an infinite dimensional Banach space.

In this paper we only consider the spaces $\mathcal{L}(^kE : E), \mathcal{L}_s(^kE : E)$ and $\mathcal{P}(^kE : E)$. Let

$$\Pi(E^k) = \{ \ (x_1, \dots, x_k, x^*) \ : \ x^*(x_j) = 1, \ x_j \in S_E, \ x^* \in S_{E^*}, \ j = 1, \cdots, k\}.$$

The numerical range of $A \in \mathcal{L}(^kE : E)$ is defined by

$$W(A) := \ \{ \ x^*(A(x_1, \dots, x_k)) \ : \ (x_1, \dots, x_k, x^*) \in \Pi(E^k)\}$$

and the numerical radius of $A \in \mathcal{L}(^kE : E)$ is defined by

$$v(A) := \sup \ \{ \ |x^*(A(x_1, \dots, x_k))| \ : \ (x_1, \dots, x_k, x^*) \in \Pi(E^k)\}.$$

Similarly, for each $P \in \mathcal{P}(^kE : E)$, the numerical range of P is defined by

$$W(P) := \{x^*(Px) : (x, x^*) \in \Pi(E^1)\}$$

and the numerical radius of P is defined by

$$v(P) := \sup \ \{|\lambda| : \lambda \in W(P)\}.$$

Clearly we have $v(A) \leq \|A\|, v(A_s) \leq \|A_s\|$ and $v(\widehat{A}) \leq \|\widehat{A}\|$, for any A in $\mathcal{L}(^kE : E)$. It is obvious that

$$(*) \qquad v(\widehat{A}) \leq v(A_s) \leq v(A) \qquad (A \in \mathcal{L}(^kE : E))$$

as in case of norms of them. The following example shows that the inequalities in $(*)$ can be strict: In fact, we define a continuous 2-linear map $A : l_1 \to l_1$ by

$$A(x, y) = (\frac{1}{2}x_1y_1 + 2x_1y_2)e_1 + (-\frac{1}{2}x_2y_2 - x_1y_2)e_2$$

for any $x = (x_i), y = (y_i) \in l_1$, where $e_1 = (1, 0, 0, \cdots)$ and $e_2 = (0, 1, 0, 0, \cdots)$. Then we have

$$A_s(x, y) = (\frac{1}{2}x_1y_1 + x_1y_2 + x_2y_1)e_1 + (-\frac{1}{2}x_2y_2 - \frac{1}{2}x_1y_2 - \frac{1}{2}x_2y_1)e_2$$

and

$$\widehat{A}(x) = (\frac{1}{2}x_1^2 + 2x_1x_2)e_1 + (-\frac{1}{2}x_2^2 - x_1x_2)e_2.$$

It is not difficult to show that $v(\widehat{A}) = \frac{1}{2}, \|\widehat{A}\| = 1, v(A_s) = \frac{3}{2} = \|A_s\|$ and $v(A) = 3 = \|A\|$. Thus $v(\frac{\widehat{A}}{\|\widehat{A}\|}) < v(\frac{A_s}{\|A_s\|}) = v(\frac{A}{\|A\|})$. Note that $\|\widehat{A}\| < \|A_s\| < \|A\|$ and $v(\widehat{A}) < v(A_s) < v(A)$.

In [CGKM2] the $k-$th polynomial numerical index of E, the constant $n_p^{(k)}(E)$ is defined by

$$n_p^{(k)}(E) := \inf\ \{v(P) : P \in S_{\mathcal{P}(^kE:E)}\}.$$

Clearly $0 \leq n_p^{(k)}(E) \leq 1$ We refer to [BD, CK, CGKM1–3, DMPW, FMP, K, KMM, L, LMP, Lu, MP] for general information and background on the theory of numerical index of Banach spaces.

In connection to $n_p^{(k)}(E)$, very recently the author introduced [K] the new concepts of the k-th numerical index and k-th symmetric numerical index of E, generalizing to k-linear and symmetric k-linear maps, respectively the "classical" numerical index defined by G. Lumer [Lu] in the sixties for linear operators. In [K] the k-th *multilinear numerical index* of E is defined by

$$n_m^{(k)}(E) := \inf\ \{v(A) : A \in S_{\mathcal{L}(^kE:E)}\}$$

and the k-th *symmetric multilinear numerical index* of E is defined by

$$n_s^{(k)}(E) := \inf\ \{v(A) : A \in S_{\mathcal{L}_s(^kE:E)}\}.$$

Clearly $0 \leq n_m^{(k)}(E) \leq 1, 0 \leq n_s^{(k)}(E) \leq 1$. Since $\mathcal{L}_s(^kE : E)$ is a closed subspace of $\mathcal{L}(^kE : E)$, we have $n_m^{(k)}(E) \leq n_s^{(k)}(E)$. Clearly $n_m^{(k)}(E)$ ($n_s^{(k)}(E)$ resp.) is the greatest constant $c \geq 0$ such that $c\|A\| \leq v(A)$ for every $A \in \mathcal{L}(^kE : E)$ ($A \in \mathcal{L}_s(^kE : E)$ resp.). Note that $n_m^{(k)}(E) > 0$ ($n_s^{(k)}(E) > 0$ resp) if and only if v and $\|\cdot\|$ are equivalent norms on $\mathcal{L}(^kE : E)$ ($\mathcal{L}_s(^kE : E)$ resp). It is easy to verify that if E_1, E_2 are isometrically isomorphic Banach spaces, then $n_m^{(k)}(E_1) = n_m^{(k)}(E_2)$ and $n_s^{(k)}(E_1) = n_s^{(k)}(E_2)$. In this paper we show that $n_I^{(k)}(E^{**}) \leq n_I^{(k)}(E)$ for $I = m, s$ and present some inequalities among $n_m^{(k)}(E), n_s^{(k)}(E)$ and $n_p^{(k)}(E)$. We also prove that if E is a strictly convex Banach space, then $n_m^{(k)}(E) = 0$ for every $k \geq 2$.

2. The inequality $n_I^{(k)}(E^{**}) \leq n_I^{(k)}(E)$ for $I = m, s$. Let E and F be Banach spaces. $L \in \mathcal{L}(^kE : F)$ has an extension $\overline{L} \in \mathcal{L}(^kE^{**} : F^{**})$ to the bidual E^{**} of E, which is called an extension of L by the Aron-Berner method (see [AB]). In fact, an extension of L, say, $\overline{L}$ is defined in the following way: We first start with the complex-valued bounded k-linear map $L \in \mathcal{L}(^kE)$. We can extend L to an k-linear form $\overline{L}$ on the bidual E^{**} in such a way that for each fixed j, $1 \leq j \leq k$ and for each fixed $x_1, \ldots, x_{j-1} \in E$ and $z_{j+1}, \ldots, z_m \in E^{**}$, the linear form

$$z \to \overline{L}(x_1, \ldots, x_{j-1}, z, z_{j+1}, \ldots, z_k), \quad z \in E^{**},$$

is weak-star continuous. By this weak-star continuity L can be extended to an k-linear form $\overline{L}$ on E^{**}, beginning with the last variable and working backwards to the first. It is not difficult to show that $\|L\| = \|\overline{L}\|$. It is also worth to remark that $\overline{L}$ is not symmetric in general and that there may exist $k!$ extensions of L to E^{**} by the Aron-Berner method. Next, for a vector-valued k-linear map $L \in \mathcal{L}({}^kE : F)$, an extension by the Aron-Berner method $\overline{L} \in \mathcal{L}({}^kE^{**} : F^{**})$ is defined as follows: given $z_1, \cdots, z_k \in E^{**}$ and $w \in F^*$,

$$\overline{L}(z_1, \cdots, z_k)(w) = \overline{w \circ L}(z_1, \cdots, z_k).$$

For $x \in E$, we define $\delta_x : E^* \to \mathbb{C}$ by $\delta_x(x^*) = x^*(x)$ for each $x^* \in E^*$. Then $\delta_x \in E^{**}$. Let us notice that a continuous k-linear map $L : E^k \to F$ is separately (w*,w*)-continuous on E^k.

THEOREM 2.1. *Let $P \in \mathcal{P}({}^kE : E)$ $(k \in \mathbb{N})$ with $\overline{\check{P}} \in \mathcal{L}_s({}^kE^{**} : E^{**})$, where $\check{P}$ is the associated continuous symmetric k-linear map to P. Then P is (w*, w*)-continuous on E if and only if $\check{P}$ is separately (w*, w*)-continuous on E^k.*

Proof. $(\Leftarrow)$: Let $x_0^{''} \in E^{**}$ and let $(x_\alpha)_\Gamma$ be a net in E such that $(x_\alpha)_\Gamma$ converges weak* to $x_0^{''}$. Then the net $(\ P(x_\alpha)\ )_\Gamma = (\ \check{P}(x_\alpha, \cdots, x_\alpha)\ )_\Gamma$ converges weak* to $\overline{\check{P}}(x_0^{''}, \cdots, x_0^{''}) = \overline{\check{P}}(x_0^{''}, \cdots, x_0^{''}) = \overline{P}(x_0^{''})$.

$(\Rightarrow)$: *Claim*: $\overline{\check{P}} = \check{\overline{P}}$.

For any $x^{''} \in E^{**}$, we have

$$\widehat{\overline{\check{P}}}(x^{''}) = \overline{P}(x^{''}) = \overline{\check{P}}(x_0^{''}, \cdots, x_0^{''}) = \widehat{\check{\overline{P}}}(x_0^{''}).$$

Since by the hypothesis $\overline{\check{P}}$ is a symmetric k-linear map, by the uniqueness of the associated symmetric k-linear map to the polynomial $\overline{P}$, we complete the proof of claim. Let $x_1^{''}, \cdots, x_k^{''} \in E^{**}$ and let $(x_{\alpha_1}^{(1)})_{\Gamma_1}, \cdots, (x_{\alpha_k}^{(k)})_{\Gamma_k}$ be nets in E such that for each $i = 1, \cdots, k$, $(x_{\alpha_i}^{(i)})_{\Gamma_i}$ converges weak* to $x_i^{''}$. Then the net $(\ \epsilon_1 x_{\alpha_1}^{(1)} + \cdots + \epsilon_k x_{\alpha_k}^{(k)}\ )_{\Gamma_1, \cdots, \Gamma_k}$ converges weak* to $\epsilon_1 x_1^{''} + \cdots + \epsilon_k x_k^{''}$ for any $\epsilon_1, \cdots, \epsilon_k \in \mathbb{K}$. By the polarization formula, the net

$$\begin{aligned} &(\ \check{P}(x_{\alpha_1}^{(1)}, \cdots, x_{\alpha_k}^{(k)})\ )_{\Gamma_1, \cdots, \Gamma_k} \\ &\qquad = (\ \frac{1}{2^k k!} \sum_{\epsilon_1, \cdots, \epsilon_k = \pm 1} \epsilon_1 \cdots \epsilon_k\ P(\epsilon_1 x_{\alpha_1}^{(1)} + \cdots + \epsilon_k x_{\alpha_k}^{(k)})\ )_{\Gamma_1, \cdots, \Gamma_k} \end{aligned}$$

converges weak* to

$$\frac{1}{2^k k!} \sum_{\epsilon_1, \cdots, \epsilon_k = \pm 1} \epsilon_1 \cdots \epsilon_k\ \overline{P}(\epsilon_1 x_1^{''} + \cdots + \epsilon_k x_k^{''}) = \check{\overline{P}}(x_1^{''}, \cdots, x_k^{''}) = \overline{\check{P}}(x_1^{''}, \cdots, x_k^{''}).$$

□

Note that Corollary 2.15 of [CDKM2] shows that $n_p^{(k)}(E^{**}) \le n_p^{(k)}(E)$. For the k-multilinear numerical index and symmetric multilinear numerical index cases, we get the analogous results, respectively.

Theorem 2.2. *Let E be a Banach space. Let $A \in \mathcal{L}({}^kE^{**} : E^{**})$ $(k \in \mathbb{N})$ be such that $A|_E$ is separately* (w*, w*)*-continuous. Let*

$$lW(A) := \{A(\delta_{x_1}, \cdots, \delta_{x_k})(x') : (x_1, \cdots, x_k, x') \in \Pi(E^k)\}.$$

Then $lW(A) \subset W(A) \subset \overline{lW(A)}$, so $\overline{lW(A)} = \overline{W(A)}$.

Proof. We may assume $\|A\| = 1$. Clearly $lW(A) \subset W(A)$. Let $0 < \epsilon < 1$. By the uniform continuity of A on $(B_{E^{**}})^k$ there is a $0 < \delta < \frac{\epsilon}{3}$ such that for $(y_1'', \cdots, y_k''), (z_1'', \cdots, z_k'') \in (B_{E^{**}})^k$ with $\|y_i'' - z_i''\| < \delta$ for all $i = 1, \cdots, k$ we have

$$(*)\quad \|A(y_1'', \cdots, y_k'') - A(z_1'', \cdots, z_k'')\| < \frac{\epsilon}{3}.$$

Let

$$LW(A) := \{A(x_1'', \cdots, x_k'')(x') : (x_1'', \cdots, x_k'', \delta_{x'}) \in \Pi((E^{**})^k)\}.$$

Claim 1: $W(A) \subset \overline{LW(A)}$.

Let $\lambda \in W(A)$. Then $\lambda = x_0'''(A(x_1'', \cdots, x_k''))$ for some $(x_1'', \cdots, x_k'', x_0''') \in \Pi((E^{**})^k)$. Since, by Goldstine's theorem (B_{E^*} is w*-dense in $B_{E^{***}}$), there exists $x_0' \in B_{E^*}$ such that

$$|\delta_{x_0'}(A(x_1'', \cdots, x_k'')) - x_0'''(A(x_1'', \cdots, x_k''))| = |A(x_1'', \cdots, x_k'')(x_0') - \lambda| < \delta$$

and for all $i = 1, \cdots, k$

$$|\delta_{x_0'}(x_i'') - x_0'''(x_i'')| = |x_i''(x_0') - 1| < \frac{\delta^2}{4}.$$

By the Bishop-Phelps-Bollobás theorem [B] there is $y_0' \in B_{E^*}$ and $y_1'', \cdots, y_k'' \in B_{E^{**}}$ such that $\|x_0' - y_0'\| < \delta$, and for all $i = 1, \cdots, k$, $\|x_i'' - y_i''\| < \delta$ and $(y_1'', \cdots, y_k'', \delta_{y_0'}) \in \Pi((E^{**})^k)$. Then $A(y_1'', \cdots, y_k'')(y_0') \in LW(A)$. It follows that

$$\begin{aligned}
& |\lambda - A(y_1'', \cdots, y_k'')(y_0')| \\
\le\ & |\lambda - A(x_1'', \cdots, x_k'')(x_0')| + |A(x_1'', \cdots, x_k'')(x_0') - A(x_1'', \cdots, x_k'')(y_0')| \\
+\ & |A(x_1'', \cdots, x_k'')(y_0') - A(y_1'', \cdots, y_k'')(y_0')| \\
<\ & \delta + \|A(x_1'', \cdots, x_k'')\|\ \|x_0' - y_0'\| + \|A(x_1'', \cdots, x_k'') - A(y_1'', \cdots, y_k'')\| \\
<\ & \delta + \delta + \frac{\epsilon}{3} < \epsilon,
\end{aligned}$$

showing $\lambda \in \overline{LW(A)}$. Thus $W(A) \subset \overline{LW(A)}$.

Claim 2: $LW(A) \subset \overline{lW(A)}$.

Let $\beta \in LW(A)$. Then $\beta = A(x_1'', \cdots, x_k'')(x_0')$ for some $(x_1'', \cdots, x_k'', \delta_{x_0'}) \in \Pi((E^{**})^k)$. Let $0 < \epsilon < 1$. By Goldstine's theorem (B_E is w*-dense in $B_{E^{**}}$), there exist k-nets $(x_{\alpha_1}^{(1)})_{\Gamma_1}, \cdots, (x_{\alpha_k}^{(k)})_{\Gamma_k}$ in B_E such that $\delta_{x_{\alpha_i}^{(i)}}$ converges weak* to x_i'' for each $i = 1, \cdots, k$. Then $\delta_{x_0'}(\delta_{x_{\alpha_i}^{(i)}}) = x_0'(x_{\alpha_i}^{(i)})$ converges to $\delta_{x_0'}(x_i'') = x_i''(x_0') = 1$ for

each $i = 1, \cdots, k$. Let $B_0 = \delta_{x_0'} \circ A \in \mathcal{L}(^k E^{**})$. Since $A \in \mathcal{L}(^k E^{**} : E^{**})$ is separately (w*, w*)-continuous, $B_0(\delta_{x^{(1)}_{\alpha_1}}, \cdots, \delta_{x^{(k)}_{\alpha_k}}) = A(\delta_{x^{(1)}_{\alpha_1}}, \cdots, \delta_{x^{(k)}_{\alpha_k}})(x_0')$ converges to $B_0(x_1'', \cdots, x_k'') =$
$A(x_1'', \cdots, x_k'')(x_0') = \beta$. Choose $\tilde{\alpha}_1 \in \Gamma_1, \cdots, \tilde{\alpha}_k \in \Gamma_k$ such that

$$|\beta - A(x^{(1)}_{\tilde{\alpha}_1}, \cdots, x^{(k)}_{\tilde{\alpha}_k})(x_0')| < \delta \text{ and } |x_0'(x^{(i)}_{\tilde{\alpha}_i}) - 1| < \frac{\delta^2}{4} \text{ for all } i = 1, \cdots, k.$$

By the Bishop-Phelps-Bollobás theorem, there is $(y^{(1)}_{\alpha_1}, \cdots, y^{(k)}_{\alpha_k}, y_0') \in \Pi(E^k)$ such that $\|x_0' - y_0'\| < \delta$ and $\|x^{(i)}_{\tilde{\alpha}_i} - y^{(i)}_{\alpha_i}\| < \delta$ for all $i = 1, \cdots, k$.

Then $A(y^{(1)}_{\alpha_1}, \cdots, y^{(k)}_{\alpha_k})(y_0') \in lV(P)$. We have

$$\begin{aligned}
& |\beta - A(y^{(1)}_{\alpha_1}, \cdots, y^{(k)}_{\alpha_k})(y_0')| \\
\leq\ & |\beta - A(x^{(1)}_{\tilde{\alpha}_1}, \cdots, x^{(k)}_{\tilde{\alpha}_k})(x_0')| + |A(x^{(1)}_{\tilde{\alpha}_1}, \cdots, x^{(k)}_{\tilde{\alpha}_k})(x_0') - A(y^{(1)}_{\alpha_1}, \cdots, y^{(k)}_{\alpha_k})(x_0')| \\
+\ & |A(y^{(1)}_{\alpha_1}, \cdots, y^{(k)}_{\alpha_k})(x_0') - A(y^{(1)}_{\alpha_1}, \cdots, y^{(k)}_{\alpha_k})(y_0')| \\
<\ & \delta + \|A(x^{(1)}_{\tilde{\alpha}_1}, \cdots, x^{(k)}_{\tilde{\alpha}_k}) - A(y^{(1)}_{\alpha_1}, \cdots, y^{(k)}_{\alpha_k})\| + \|A(y^{(1)}_{\alpha_1}, \cdots, y^{(k)}_{\alpha_k})\| \, \|x_0' - y_0'\| \\
<\ & 3\delta < \epsilon \ \ (\text{by } (*)),
\end{aligned}$$

showing $\beta \in \overline{lW(A)}$. Thus $LW(A) \subset \overline{lW(A)}$. Thus by claims 1–2, $W(A) \subset \overline{lW(A)}$. □

COROLLARY 2.3. *Let E be a Banach space and $k \in \mathbb{N}$. Let $L \in \mathcal{L}(^k E : E)$. If $\overline{L}$ is an extension of L to E^{**} by the Aron-Berner method, then $\overline{V(L)} = \overline{V(\overline{L})}$. Thus $v(L) = v(\overline{L})$ and $n_m^{(k)}(E^{**}) \leq n_m^{(k)}(E)$.*

THEOREM 2.4. *Let E be a Banach space. Let $L \in \mathcal{L}(^k E : E)$ $(k \in \mathbb{N})$. If $\overline{L}$ is an extension of L to E^{**} by the Aron-Berner method, then $v(\overline{L_s}) = v(L_s) = v((\overline{L})_s)$. In particular, if L is a symmetric k-linear map, then $v(\overline{L}) = v((\overline{L})_s)$. Thus $n_s^{(k)}(E^{**}) \leq n_s^{(k)}(E)$.*

Proof. We may assume $\|L\| = 1$. Then $\|L_s\| \leq 1$. By Corollary 2.3, we have $v(\overline{L_s}) = v(L_s) \leq v((\overline{L})_s)$ because $\overline{L}$ is an extension of L. We will show the reverse inequality.

Claim: $v((\overline{L})_s) \leq v(\overline{L_s})$.

Let $0 < \epsilon < 1$. By the uniform continuity of L_s on $(B_E)^k$ there exists a $0 < \delta < \frac{\epsilon}{4}$ such that for $(y_1, \cdots, y_k), (z_1, \cdots, z_k) \in (B_E)^k$ with $\|y_i - z_i\| < \delta$ $(i = 1, \cdots, k)$, we have

$$(*) \quad \|L_s(y_1, \cdots, y_k) - L_s(z_1, \cdots, z_k)\| < \frac{\epsilon}{4}.$$

Let $(x_1'', \cdots, x_k'', x_0''') \in \Pi((E^{**})^k)$. By the Goldstine theorem, there exist a net $(x_\beta')_\Lambda$ in B_{E^*} and nets $(x^{(1)}_{\alpha_1})_{\alpha_1 \in \Gamma_1}, \cdots, (x^{(k)}_{\alpha_k})_{\alpha_k \in \Gamma_k}$ in B_E such that $(x^{(i)}_{\alpha_i})_{\alpha_i \in \Gamma_i}$

converges w* to $x_i^{''}$ for all $i = 1, \cdots, k$ and $(x_\beta^{'})_\Lambda$ converges w* to $x_0^{'''}$. Then we have, for each $i = 1, \cdots, k$,

$$(**) \quad \lim_\beta \lim_{\alpha_i} x_\beta^{'}(x_{\alpha_i}^{(i)}) = \lim_\beta x_\beta^{'}(x_i{''}) = x_0^{'''}(x_i{''}) = 1.$$

Since $(\overline{L})_s(x_1^{''}, \cdots, x_k^{''}) \in E^{**}$ and $(x_\beta^{'})_\Lambda$ converges weak* to $x_0^{'''}$, we have

$$\lim_\beta (\overline{L})_s(x_1^{''}, \cdots, x_k^{''})(x_\beta^{'}) = x_0^{'''}((\overline{L})_s(x_1^{''}, \cdots, x_k^{''})).$$

Choose $\beta_0 \in \Lambda$ such that for any $\beta \geq \beta_0$,

$$(***) \quad | \, x_0^{'''}((\overline{L})_s(x_1^{''}, \cdots, x_k^{''})) - (\overline{L})_s(x_1^{''}, \cdots, x_k^{''})(x_\beta^{'}) \, | < \delta.$$

By $(**)$ there exist $\beta_1 \in \Lambda$ with $\beta_1 \geq \beta_0$ and $\tilde{\alpha_1} \in \Gamma_1, \cdots, \tilde{\alpha_k} \in \Gamma_k$ such that for any $\alpha_i \in \Gamma_i$ with $\alpha_i \geq \tilde{\alpha_i}$ $(i = 1, \cdots, k)$, we have

$$|1 - x_{\beta_1}^{'}(x_{\alpha_i}^{(i)})| < \frac{\delta^2}{4}.$$

Since $(\overline{L})_s$ is separately (w*, w*)-continuous, there exist $\tilde{\tilde{\alpha}}_1 \in \Gamma_1$ with $\tilde{\tilde{\alpha}}_1 \geq \tilde{\alpha}_1$, $\cdots$, $\tilde{\tilde{\alpha}}_k \in \Gamma_k$ with $\tilde{\tilde{\alpha}}_k \geq \tilde{\alpha}_k$ such that

$$\begin{aligned} (****) \qquad & |(\overline{L})_s(x_1^{''}, \cdots, x_k^{''})(x_{\beta_1}^{'}) - (\overline{L})_s(x_{\tilde{\tilde{\alpha}}_1}^{(1)}, \cdots, x_{\tilde{\tilde{\alpha}}_k}^{(k)})(x_{\beta_1}^{'})| \\ = \; & |(\overline{L})_s(x_1^{''}, \cdots, x_k^{''})(x_{\beta_1}^{'}) - L_s(x_{\tilde{\tilde{\alpha}}_1}^{(1)}, \cdots, x_{\tilde{\tilde{\alpha}}_k}^{(k)})(x_{\beta_1}^{'})| < \delta. \end{aligned}$$

By the Bishop-Phelps-Bollobás theorem, there is $y_0^{'} \in B_{E^*}$ and $y_1, \cdots, y_k \in B_E$ such that $(y_1, \cdots, y_k, y_0^{'}) \in \Pi(E^k)$, $\|x_{\beta_1}^{'} - y_0^{'}\| < \delta$ and for all $i = 1, \cdots, k$, $\|x_{\tilde{\tilde{\alpha}}_i}^{(i)} - y_i\| < \delta$. Then $L_s(y_1, \cdots, y_k)(y_0^{'}) \in V(L_s)$. Then we have

$$\begin{aligned} & | \, x_0^{'''}((\overline{L})_s(x_1^{''}, \cdots, x_k^{''})) - L_s(y_1, \cdots, y_k)(y_0^{'}) \, | \\ \leq \; & | \, x_0^{'''}((\overline{L})_s(x_1^{''}, \cdots, x_k^{''})) - (\overline{L})_s(x_1^{''}, \cdots, x_k^{''})(x_{\beta_1}^{'}) \, | \\ + \; & | \, (\overline{L})_s(x_1^{''}, \cdots, x_k^{''})(x_{\beta_1}^{'}) - (\overline{L})_s(L_s(x_{\tilde{\tilde{\alpha}}_1}^{(1)}, \cdots, x_{\tilde{\tilde{\alpha}}_k}^{(k)}))(x_{\beta_1}^{'})| \\ + \; & |L_s(x_{\tilde{\tilde{\alpha}}_1}^{(1)}, \cdots, x_{\tilde{\tilde{\alpha}}_k}^{(k)})(x_{\beta_1}^{'}) - L_s(y_1, \cdots, y_k)(x_{\beta_1}^{'})| \\ + \; & |L_s(y_1, \cdots, y_k)(x_{\beta_1}^{'}) - L_s(y_1, \cdots, y_k)(y_0^{'}) \, | \\ < \; & \delta + \delta + \| \, L_s(x_{\tilde{\tilde{\alpha}}_1}, \cdots, x_{\tilde{\tilde{\alpha}}_k}) - L_s(y_1, \cdots, y_k) \, \| \quad (\text{by } (***) \text{ and } (****)) \\ + \; & \|L_s(y_1, \cdots, y_k) \, \| \, \|x_{\beta_1}^{'} - y_0^{'}\| \\ < \; & 2\delta + \frac{\epsilon}{4} + \|x_{\beta_1}^{'} - y_0^{'}\| \quad (\text{by } (*)) \\ < \; & \epsilon, \end{aligned}$$

which shows the claim. Note that $\|L\| = \|(\overline{L})_s\| = \|\overline{L}\|$ for all $L \in \mathcal{L}_s({}^kE : E)$. Thus $n_s^{(k)}(E^{**}) \leq n_s^{(k)}(E)$. □

3. Inequalities between $n_s^{(k)}(E)$ and $n_p^{(k)}(E)$ and estimates for $n_I^{(k)}(E)$ for $I = m, s, p$.

LEMMA 3.1. ([CK], Theorem 3.5) *For each $k \in \mathbb{N}$ and each $P \in \mathcal{P}(^kE : F)$, we have*

$$v(P) \leq v(\check{P}) \leq \frac{\sum_{j=1}^{k} j^k \ {}_kC_j}{k!} \ v(P),$$

where ${}_kC_j = \frac{k!}{j!(k-j)!}$.

Using Lemma 3.1, we obtain some inequalities between $n_s^{(k)}(E)$ and $n_p^{(k)}(E)$.

THEOREM 3.2. *For every Banach space E and every $k \in \mathbb{N}$ we have*

$$\frac{k!}{k^k} \ n_p^{(k)}(E) \leq n_s^{(k)}(E) \leq \frac{\sum_{j=1}^{k} j^k \ {}_kC_j}{k!} \ n_p^{(k)}(E).$$

Proof. Put $M = \frac{\sum_{j=1}^{k} j^k \ {}_kC_j}{k!}$. It follows:

$$\begin{aligned} n_s^{(k)}(E) &= \inf_{A\in\mathcal{L}_s(^kE:E), A\neq 0} v(\frac{A}{\|A\|}) = \inf_{A\in\mathcal{L}_s(^kE:E), A\neq 0} \frac{1}{\|A\|} \ v(A) \\ &\leq \inf_{A\in\mathcal{L}_s(^kE:E), A\neq 0} \frac{1}{\|A\|} \ M \ v(\widehat{A}) = \inf_{A\in\mathcal{L}_s(^kE:E), A\neq 0} \frac{\|\widehat{A}\|}{\|A\|} \ M \ v(\frac{\widehat{A}}{\|\widehat{A}\|}) \\ &\qquad\qquad \text{(by Lemma 3.1)} \\ &\leq \quad M \inf_{P\in\mathcal{P}(^kE:E), P\neq 0} v(\frac{P}{\|P\|}) = M \ n_p^{(k)}(E). \end{aligned}$$

On the other hand we have

$$\begin{aligned} n_p^{(k)}(E) &= \inf_{P\in\mathcal{P}(^kE:E), P\neq 0} v(\frac{P}{\|P\|}) = \inf_{P\in\mathcal{P}(^kE:E), P\neq 0} \frac{1}{\|P\|} \ v(P) \\ &\leq \inf_{P\in\mathcal{P}(^kE:E), P\neq 0} \frac{1}{\|P\|} \ v(\check{P}) = \inf_{P\in\mathcal{P}(^kE:E), P\neq 0} \frac{\|\check{P}\|}{\|P\|} \ v(\frac{\check{P}}{\|\check{P}\|}) \\ &\leq \inf_{A\in\mathcal{L}_s(^kE:E), A\neq 0} \frac{k^k}{k!} \ v(\frac{A}{\|A\|}) = \frac{k^k}{k!} \ n_s^{(k)}(E). \end{aligned}$$

□

COROLLARY 3.3. *We have $n_s^{(k)}(E) = 0$ if and only if $n_p^{(k)}(E) = 0$ for a Banach space E.*

Example 2.6 of [KMM] shows that there exists a real Banach space X_0 such that $0 = n_p^{(k)}(X_0^{**}) < n_p^{(k)}(X_0)$ for all $k = 1, 2, \cdots$. By Theorem 3.2, we have $0 = n_s^{(k)}(X_0^{**}) < n_s^{(k)}(X_0)$ for all $k = 1, 2, \cdots$. For $E = X_0$, the inequality of Theorem 2.4 is strict.

Theorem 3.4. *Let $E = c_0, l_\infty, l_1, C(K)$ (K is a scattered compact Hausdorff space), A_D (A_D is the disc algebra) and $k \in \mathbb{N}$. Then $n_m^{(k)}(E) = n_s^{(k)}(E) = 1$.*

Proof. By [CK, Theorem 3.1(i), Theorem 3.2], [CGKM1, Theorem 3.3] and [CGKM2, Theorem 3.2] it follows that $v(A) = \|A\|$ for every $A \in \mathcal{L}(^kE : E)$. □

Theorem 3.5. *If $n_p^{(k)}(E) = 1$, then $n_s^{(k)}(E) = 1 = n_m^{(k)}(E)$.*

Proof. It follows from the fact of $E = \mathbb{R}$ by a result of [L]. □

Note that the converse of Theorem 3.5 is not true in general since $n_p^{(2)}(l_1) = \frac{1}{2}, n_s^{(2)}(l_1) = 1 = n_m^{(2)}(l_1)$ by Theorem 3.4 and Corollary 2.5 of [KMM].

We get some lower bound for $n_s^{(k)}(E)$ as follows:

Theorem 3.6. *For every complex Banach space E and every $k \geq 2$ we have*

$$k^{\frac{k^2}{1-k}} \quad k! \leq n_s^{(k)}(E).$$

Proof. It follows from the fact that $k^{\frac{k}{1-k}} \leq n_p^{(k)}(E)$ of [CGKM2, Theorem 2.3] and Theorem 3.2. □

It is obvious that $v(\hat{A}) \leq v(A_s) \leq v(A)$ for any A in $\mathcal{L}(^kE : E)$. The following shows that these three quantities are equal in case E is a strictly convex Banach space.

Theorem 3.7. *Let $k \in \mathbb{N}$ and E a strictly convex Banach space. Then*
(1) $v(\widehat{A}) = v(A_s) = v(A)$ for each $A \in \mathcal{L}(^kE : E)$;
(2) $n_m^{(k)}(E) = n_m^{(k)}(E^{**}) = 0$ for any $k \geq 2$;
(3) $n_s^{(k)}(E) \leq n_p^{(k)}(E)$.

Proof. (1): $(x_1, \ldots, x_k, x^*) \in \Pi(E^k)$. then we have, for $j = 2, \cdots, k$,

$$1 = |x^*(\frac{x_1 + x_j}{2})| \leq \|\frac{x_1 + x_j}{2}\| \leq 1,$$

so $\|\frac{x_1+x_j}{2}\| = 1$, thus, by strict convexity of E, we have $x_1 = x_j$ for all $j = 2, \cdots, k$. Let $A \in \mathcal{L}(^kE : E)$. Since $v(\hat{A}) \leq v(A_s) \leq v(A)$, it is enough to show that $v(A) \leq v(\widehat{A})$. It follows that

$$\begin{aligned} v(A) &= \sup_{(x_1,\ldots,x_k,x^*)\in\Pi(E^k)} |x^*(A(x_1, \ldots, x_k))| \\ &= \sup_{(x_1,x^*)\in\Pi(E^1)} |x^*(A(x_1, \ldots, x_1))| \\ &= \sup_{(x_1,x^*)\in\Pi(E^1)} |x^*(\widehat{A}(x_1))| = v(\widehat{A}). \end{aligned}$$

Thus $v(\widehat{A}) = v(A)$.

(2): *Claim:* $n_m^{(2)}(E) = 0$.

Let $\{u, v\}$ be a linearly independent subset of S_E and $w \in S_E$. By the Hahn-Banach theorem there exist x^*, y^* in S_E^* such that $x^*(u) = 1 = y^*(v)$ and $x^*(v) = 0 = y^*(u)$. We define a continuous bilinear mapping $A_0 \in \mathcal{L}(^2E : E)$ by

$$A_0(x, y) = (x^*(x)y^*(y) - x^*(y)y^*(x))w \text{ for any } x, y \in E.$$

Then $\widehat{A_0} = 0$, so $v(\widehat{A_0}) = 0$. Since $\|A_0(u, v)\| = 1$ we have $\|A_0\| \geq 1$. By (1),

$$0 \leq n_m^{(2)}(E) \leq v(\frac{A_0}{\|A_0\|}) = \frac{v(A_0)}{\|A_0\|} = \frac{v(\widehat{A_0})}{\|A_0\|} = 0.$$

By Theorem 2.1, we have $n_m^{(k)}(E) = 0$. By Corollary 2.3, we have

$$0 \leq n_m^{(k)}(E^{**}) \leq n_m^{(k)}(E) = 0.$$

(3): Let $B \in \mathcal{L}_s(^kE : E)$ with $\|B\| = 1$. Let $P = \widehat{B}$. Then $0 < \|P\| \leq 1$. Since $v(P) = v(B)$, by (1), it follows that

$$(*) \quad v(B) = v(P) \leq \frac{v(P)}{\|P\|} = v(\frac{P}{\|P\|}).$$

By taking infimum over $B \in \mathcal{L}_s(^kE : E)$ with $\|B\| = 1$ in $(*)$, we see that $n_s^{(k)}(E) \leq n_p^{(k)}(E)$. $\square$

COROLLARY 3.8. *Let $k \in \mathbb{N}$ and $1 < p < \infty$. Suppose that (X, Ω, μ) is a measure space. If $E = L_p(\mu)$ or Hilbert space, then*

(1) $v(\widehat{A}) = v(A_s) = v(A)$ for each $A \in \mathcal{L}(^kE : E)$;

(2) $n_m^{(k)}(E) = 0$ for any $k \geq 2$;

(3) $n_s^{(k)}(E) \leq n_p^{(k)}(E)$.

EXAMPLES 3.9. In the cases of $E = l_1, l_\infty$, Theorem 3.7 is not true.

(1) We define a continuous 2-linear mapping $A : l_1 \to l_1$ by

$$A(x, y) = x_1y_2(e_1 + e_2)$$

for any $x = (x_i), y = (y_i) \in l_1$. Then we have $A_s(x, y) = \frac{x_1y_2+x_2y_1}{2}(e_1 + e_2)$ and $\widehat{A}(x) = x_1x_2(e_1 + e_2)$. It is easy to verify that $v(\widehat{A}) = \frac{1}{2} = \|\widehat{A}\|$, $v(A_s) = 1 = \|A_s\|$ and $v(A) = 2 = \|A\|$. Thus $v(\widehat{A}) < v(A_s) < v(A)$.

(2) We define a continuous 2-linear mapping $A : l_\infty \to l_\infty$ by

$$A(x, y) = x_1y_1e_1 + (x_1y_2 - x_2y_1)e_2$$

for any $x = (x_i), y = (y_i) \in l_\infty$. Then we have $A_s(x, y) = x_1y_1e_1$ and $\widehat{A}(x) = x_1^2e_1$.

Claim: $v(\widehat{A}) = v(A_s) = 1 = \|A_s\| = \|\widehat{A}\|$ and $v(A) = 2 = \|A\|$.

Indeed, it is easy to verify that $v(\widehat{A}) = v(A_s) = 1 = \|A_s\| = \|\widehat{A}\|$. We have

$$\begin{aligned} 2 &= \|-e_1 + 2e_2\| = \|A(e_1 + e_2, -e_1 + e_2)\| \leq \|A\| \\ &= \max\{\|A(x,y)\| : \ x = x_1e_1 + x_2e_2, y = y_1e_1 + y_2e_2 \in S_{l_\infty}\} \\ &\leq \max_{|x_1|=|x_2|=1=|y_1|=|y_2|} \{ \ |x_1| \ |y_1|, |x_1| \ |y_2| + |x_2| \ |y_1| \ \} \\ &\leq 2, \end{aligned}$$

showing $\|A\| = 2$. Let $x^* = (0, 1, 0, 0, \cdots) \in S_{l_\infty^*}$. Then $x^*(e_1 + e_2) = 1 = x^*(-e_1 + e_2)$, so $(e_1 + e_2, -e_1 + e_2, x^*) \in \Pi(\ell_\infty^2)$. Then we have

$$2 = |x^*(-e_1 + 2e_2)| = |x^*(A(e_1 + e_2, -e_1 + e_2))| \leq v(A) \leq \|A\| = 2,$$

so $v(A) = 2$.

THEOREM 3.10. *For every $k \in \mathbb{N}$ and every $1 < p < \infty$ we have*

$$n_s^{(k)}(\ell_p) \leq n_p^{(k)}(\ell_p) \leq \left(\frac{p-1}{k+p-1}\right)^{1-\frac{1}{p}} \left(\frac{k}{k+p-1}\right)^{\frac{k}{p}}.$$

In particular, $\lim_{k\to\infty} n_s^{(k)}(\ell_p) = \lim_{k\to\infty} n_p^{(k)}(\ell_p) = 0$.

Proof. Let $P(x) = x_2^k e_1$ for $x = (x_i) \in \ell_p$. Then $P \in \mathcal{P}(^k\ell_p : \ell_p)$ and $\|P\| = 1$. Put $f(x) = x^{p-1} \ (1 - x^p)^{\frac{k}{p}}$ for $0 \leq x \leq 1$. It is easy to show that f has its maximum $(\frac{p-1}{k+p-1})^{1-\frac{1}{p}} \ (\frac{k}{k+p-1})^{\frac{k}{p}}$ at $x = (\frac{p-1}{k+p-1})^{\frac{1}{p}}$. It follows that, by Corollary 3.8,

$$\begin{aligned} 0 &\leq n_s^{(k)}(\ell_p) \leq n_p^{(k)}(\ell_p) \leq v(P) \\ &= \sup \ \{ \ | < (y_i), P((x_i)) > | \ : \ (y_i) \in S_{\ell_q}, \ (x_i) \in S_{\ell_p}, \ \sum_{i=1}^\infty x_iy_i = 1 \ \} \\ &= \max \ \{|y_1| \ |x_2|^k \ : \ 1 = |x_1|^p + |x_2|^p = |y_1|^q + |y_2|^q = x_1y_1 + x_2y_2 \ \} \\ &= \max \ \{|y_1| \ |x_2|^k \ : \ y_1 = x_1^{p-1}, \ 1 = |x_1|^p + |x_2|^p \ \} \\ &= \max_{0\leq x\leq 1} \ \{ \ x^{p-1}(1 - x^p)^{\frac{k}{p}} \ \} = \left(\frac{p-1}{k+p-1}\right)^{1-\frac{1}{p}} \left(\frac{k}{k+p-1}\right)^{\frac{k}{p}} \\ &\leq \left(\frac{p-1}{k+p-1}\right)^{1-\frac{1}{p}} \to 0 \text{ as } k \to \infty, \end{aligned}$$

which completes the proof. □

COROLLARY 3.11. *Let H be a* real *Hilbert space of dimension greater than 1 and $k \in \mathbb{N}$. Then $n_m^{(k)}(H) = n_s^{(k)}(H) = n_p^{(k)}(H) = 0$.*

Proof. Since $n_p^{(1)}(H) = 0$, by Proposition 2.5 of [CGKM2], $n_p^{(k)}(H) = 0$. By Corollary 3.8 we have $0 \leq n_m^{(k)}(H) \leq n_s^{(k)}(H) \leq n_p^{(k)}(H) = 0$. □

COROLLARY 3.12. *Let H be a complex Hilbert space of dimension greater than 1. Then $n_p^{(2)}(H) = n_s^{(2)}(H) \leq \frac{1}{2}$.*

Proof. We claim that $\|\check{P}\| = \|P\|$ for each $P \in \mathcal{P}(^2H : H)$. Let $P \in \mathcal{P}(^2H : H)$. It is enough to show that $\|\check{P}\| \leq \|P\|$. Since $\check{P}(x, y) = \frac{1}{4}P(x + y) - \frac{1}{4}P(x - y)$, we have for $x, y \in B_H$, by the parallelogram identity,

$$\|\check{P}(x, y)\| \leq \frac{1}{4}\|P\|(\|x + y\|^2 + \|x - y\|^2),$$

showing $\|\check{P}\| = \|P\|$. Thus we have $n_s^{(2)}(H) = n_p^{(2)}(H)$. Since $n_p^{(1)}(H) \leq \frac{1}{2}$, by Proposition 2.5 of [CGKM2], $n_s^{(2)}(H) = n_p^{(2)}(H) \leq \frac{1}{2}$. □

For a Banach space E and $k \in \mathbb{N}$, we define [K]

$$\mathbb{K}(k : E) := \inf \ \{M > 0 : \ \|A\| \leq M \ \|\widehat{A}\| \text{ for every } A \in \mathcal{L}_s(^kE) \ \}.$$

It is well-known that $1 \leq \mathbb{K}(k : E) \leq \frac{k^k}{k!}$. By Lemma 3.1 of [K], we have

$$\begin{aligned} \mathbb{K}(k : E) &= \inf \ \{ \ M > 0 : \ \|A\| \leq M \ \|\widehat{A}\| \text{ for every A } \in \mathcal{L}_s(^kE : E) \ \} \\ &= \frac{1}{\inf \ \{ \ \|\widehat{A}\| : \ A \in \mathcal{L}_s(^kE : E), \|A\| = 1 \ \}}. \end{aligned}$$

LEMMA 3.13. ([K], Theorem 3.4) *Let $k \in \mathbb{N}$. Suppose that E is a Banach space such that $v(\check{P}) = v(P)$ for each $P \in \mathcal{P}(^kE : E)$, where $\check{P}$ is the associated continuous symmetric k-linear mapping to P. Then $n_p^{(k)}(E) \leq \mathbb{K}(k : E) \ n_s^{(k)}(E)$.*

LEMMA 3.14. ([S], Theorem 3) *Let $k \geq 2$.*
Then: (1) For $1 \leq p \leq \frac{k}{k-1}$ we have

$$\mathbb{K}(k : L_p(\mu)) \leq \frac{k^{\frac{k}{p}}}{k!};$$

(2) for p≥k we have

$$\mathbb{K}(k : L_p(\mu)) \leq \frac{k^{\frac{k}{q}}}{k!},$$

where $q > 1$ is the real number such that $\frac{1}{p} + \frac{1}{q} = 1$.

THEOREM 3.15. *Let $k \geq 2$.*
Then: (1) For $1 \leq p \leq \frac{k}{k-1}$ we have

$$n_p^{(k)}(L_p(\mu)) \leq \frac{k^{\frac{k}{p}}}{k!} \ n_s^{(k)}(L_p(\mu));$$

(2) for p≥k we have

$$n_p^{(k)}(L_p(\mu)) \leq \frac{k^{\frac{k}{q}}}{k!} \ n_s^{(k)}(L_p(\mu)),$$

where $q > 1$ is the real number such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. It follows from Lemmas 3.13–14. □

COROLLARY 3.16. *Let $k \in \mathbb{N}$ be a power of 2. Then $n_s^{(k)}(L_2(\mu)) = n_p^{(k)}(L_2(\mu))$ for a complex $L_2(\mu)$ space.*

Proof. By a result of Harris [H], $\mathbb{K}(k : L_2(\mu)) = 1$ for a complex $L_2(\mu)$ space. Hence $\|A\| = \|\widehat{A}\|$ for any $A \in \mathcal{L}(^k L_2(\mu) : L_2(\mu))$. By Theorem 3.7, $v(A) = v(\widehat{A})$ for any $A \in \mathcal{L}(^k L_2(\mu) : L_2(\mu))$, which completes the proof. □

COROLLARY 3.17. *Let H be a separable Hilbert space of dimension greater than 1. Let $k \in \mathbb{N}$ be a power of 2. Then $n_s^{(k)}(H) = n_p^{(k)}(H)$.*

Proof. By the Riesz-Fischer theorem, H is isometrically isometric to l_2. If H is a real Hilbert space, by Corollary 3.11, we have

$$n_s^{(k)}(H) = 0 = n_p^{(k)}(H).$$

If H is a complex Hilbert space, by Corollary 3.16, we have

$$n_s^{(k)}(H) = n_s^{(k)}(l_2) = n_p^{(k)}(l_2) = n_p^{(k)}(H).$$ □

PROPOSITION 3.18. *Let $I = m, s, p$. Then*

(1) $n_I^{(k)}(L_p[0,1])$ is an increasing function of p over the range $1 \leq p \leq 2$;

(2) $n_I^{(k)}(L_p[0,1])$ is a decreasing function of p over the range $2 \leq p < \infty$.

Proof. Let $1 \leq p \leq r \leq 2$. Note that $L_r[0,1]$ can be embedded isometrically into $L_p[0,1]$ (see [LT], p. 139). Since if M and N are closed subspaces of a Banach space E, then $(M \oplus_{l_1} N)^* = M^* \oplus_{l_\infty} N^*$, by Theorem 3.7 of [K], $n_i^{(k)}(L_p[0,1])$ is an increasing function of p over the range $1 \leq p \leq 2$. Let $p', r' \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{r} + \frac{1}{r'}$. Then $2 \leq r' \leq p' < \infty$. Since $(L_r[0,1])^* = L_{r'}[0,1]$ can be embedded isometrically into $(L_p[0,1])^* = L_{p'}[0,1]$, by Theorem 3.7 of [K], $n_I^{(k)}(L_{p'}[0,1]) \leq n_I^{(k)}(L_{r'}[0,1])$, showing (2). □

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